1/ Math 112: Introductory Real Analysis

- The (Stone) Weierstrass theorem

(Weierstrass theorem)

Thm If f is a continuous complex function on $[a,b]$, there exists a sequence of polynomials P_n such that

$$\lim_{n\to\infty} P_n(x) = f(x) \quad \text{uniformly on } [a,b].$$

proof) Without loss of generality, we may assume that $[a,b]=[0,1]$ and that $f(0)=f(1)=0$.

Furthermore, define $f(x)$ to be 0 outside $[0,1]$, so that f is uniformly continuous on $\mathbb{R}$.

Put $Q_n(x) = c_n(1-x^2)^n \quad (n=1,2,3,\cdots)$

where c_n is chosen so that $\int_{-1}^{1} Q_n(x)\,dx = 1$.

Since
$$\int_{-1}^{1}(1-x^2)^n dx = 2\int_0^1 (1-x^2)^n dx \geq 2\int_0^{\frac{1}{\sqrt{n}}} (1-x^2)^n dx$$
$$\geq 2\int_0^{\frac{1}{\sqrt{n}}} (1-nx^2)\,dx = \frac{4}{3\sqrt{n}} > \frac{1}{\sqrt{n}},$$

we have $c_n < \sqrt{n}$.

For any $\delta > 0$, we have $Q_n(x) \leq \sqrt{n}\,(1-\delta^2)^n$ for all $\delta \leq |x| \leq 1$,

so $Q_n \to 0$ uniformly on $\delta \leq |x| \leq 1$.

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Now, set

$$P_n(x) = \int_{-1}^{1} f(x+t) Q_n(t)\,dt \quad (0 \leq x \leq 1).$$

$$= \int_{-x}^{1-x} f(x+t) Q_n(t)\,dt = \int_0^1 f(t) Q_n(t-x)\,dt,$$

which is clearly a polynomial in x.

Given $\varepsilon > 0$, choose $\delta > 0$ such that $|y-x| < \delta$ implies

$$|f(y) - f(x)| < \frac{\varepsilon}{2}.$$

Let $M = \sup |f(x)|$. We see that for $0 \leq x \leq 1$,

$$|P_n(x) - f(x)| = \left| \int_{-1}^{1} \left(f(x+t) - f(x)\right) Q_n(t)\,dt \right|$$

$$\leq \int_{-1}^{1} |f(x+t) - f(x)|\, Q_n(t)\,dt$$

$$\leq 2M \int_{-1}^{-\delta} Q_n(t)\,dt + \frac{\varepsilon}{2} \int_{-\delta}^{\delta} Q_n(t)\,dt + 2M \int_{\delta}^{1} Q_n(t)\,dt$$

$$\leq 4M\sqrt{n}\,(1-\delta^2)^n + \frac{\varepsilon}{2},$$

and the last term is $< \varepsilon$ for all large enough n, which proves the theorem. ■

~~Math 112: Introductory Real Analysis~~

~~• The Stone-Weierstrass theorem~~

Weierstrass' theorem can be generalized to a more general class of functions than just polynomials.

Def A family $\mathcal{A}$ of complex (or real) functions defined on a set E is said to be an algebra

if $\begin{cases} \text{(i) } f+g \in \mathcal{A} \\ \text{(ii) } fg \in \mathcal{A} \\ \text{(iii) } cf \in \mathcal{A} \end{cases}$ for all $f, g \in \mathcal{A}$ and $c \in \mathbb{C}$ (or $\mathbb{R}$).

If $f \in \mathcal{A}$ whenever $f_n \in \mathcal{A}$ $(n = 1, 2, 3, \cdots)$ and $f_n \to f$ uniformly on E,

we say $\mathcal{A}$ is uniformly closed.

Ex The set of all polynomials is an algebra.

Def Let $\mathcal{A}$ be a family of functions on E.

Then $\mathcal{A}$ is said to separate points if for every pair of distinct points $x_1, x_2 \in E$,

there exists $f \in \mathcal{A}$ such that $f(x_1) \neq f(x_2)$.

$\mathcal{A}$ is said to vanish at no point if for every $x \in E$, there exists $g \in \mathcal{A}$ such that $g(x) \neq 0$.

Ex The set of all even polynomials on $[-1, 1]$ do not separate points, since $f(-x) = f(x)$ for every even f.

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Thm Suppose $\mathcal{A}$ is an algebra of functions on E that separates points and vanishes at no point.

Then, for any $x_1, x_2 \in E$ with $x_1 \neq x_2$ and $c_1, c_2 \in \mathbb{R}$,

there exists $f \in \mathcal{A}$ such that $f(x_1) = c_1$ and $f(x_2) = c_2$.

proof) By assumption, $g(x_1) \neq g(x_2)$, $h(x_1) \neq 0$, $k(x_2) \neq 0$

for some $g, h, k \in \mathcal{A}$.

Put $u = gk - g(x_1)k$ and $v = gh - g(x_2)h$.

Then $u(x_1) = 0$, $\quad v(x_1) \neq 0$,

$u(x_2) \neq 0$, $\quad v(x_2) = 0$.

Therefore, $f = \dfrac{c_1}{v(x_1)} v + \dfrac{c_2}{u(x_2)} u$ has the desired properties. ■

Thm (Stone-Weierstrass theorem)

Let $\mathcal{A}$ be an algebra of real continuous functions on a compact set K.

If $\mathcal{A}$ separates points and vanishes at no point, then the uniform closure $\mathcal{B}$ of $\mathcal{A}$ (i.e. the set of all functions obtained by uniform limits of sequences of functions in $\mathcal{A}$) consists of all real continuous functions on K.

proof) Step 1: If $f \in \mathcal{B}$, then $|f| \in \mathcal{B}$

Step 2: If $f, g \in \mathcal{B}$, then $\max\{f, g\}$, $\min\{f, g\} \in \mathcal{B}$.

Step 3: Given a real continuous function f on K, a point $x \in K$, and $\varepsilon > 0$, there exists a function $g_x \in \mathcal{B}$ such that $g_x(x) = f(x)$ and

$g_x(t) > f(t) - \varepsilon$

5/ Step 4 : Given a real continuous function f on K, and $\varepsilon > 0$,

there exists $h \in \mathcal{B}$ such that

$$|h(x) - f(x)| < \varepsilon \quad \text{for all } x \in K.$$

Since $\mathcal{B}$ is uniformly continuous, this last statement is equivalent to the statement of the theorem.

proof of step 1)

Let $a = \sup_{x \in K} |f(x)|$, and let $\varepsilon > 0$ be given.

By Weierstrass theorem, there exist real numbers $c_1, \cdots, c_n$ such that

$$\left| \sum_{i=1}^{n} c_i y^i - |y| \right| < \varepsilon \quad \text{for} \quad -a \leq y \leq a.$$

Since $\mathcal{B}$ is an algebra, $g = \sum_{i=1}^{n} c_i f^i$ is in $\mathcal{B}$.

We have $\quad |g(x) - |f(x)|| < \varepsilon$ for all $x \in K$, and since $\mathcal{B}$ is uniformly closed

it follows that $|f| \in \mathcal{B}$.

proof of step 2) It follows from step 1 and the identities

$$\max\{f, g\} = \frac{f+g}{2} + \frac{|f-g|}{2},$$

$$\min\{f, g\} = \frac{f+g}{2} - \frac{|f-g|}{2}.$$

proof of step 3) Since $\mathcal{B}$ separates points and vanishes at no point,

there is $h_y \in \mathcal{B}$ such that

$$h_y(x) = f(x) \text{ and } h_y(y) = f(y).$$

By continuity of h_y, there exists an open set $U_y \ni y$ such that

$$h_y(t) > f(t) - \varepsilon \text{ for all } t \in U_y.$$

Since K is compact, there is a finite set of points $y_1, \cdots, y_n$ such that

$$K \subseteq U_{y_1} \cup \cdots \cup U_{y_n}.$$

Put $g_x = \max(h_{y_1}, \cdots, h_{y_n})$. This has the required properties.

proof Step 4) By the continuity of g_x, there exists an open set $V_x \ni x$ such that

$$g_x(t) < f(t) + \varepsilon \text{ for all } t \in V_x.$$

Since K is compact, there is a finite set of points $x_1, \cdots, x_m$ such that

$$K \subseteq V_{x_1} \cup \cdots \cup V_{x_m}.$$

Put $h = \min(g_{x_1}, \cdots, g_{x_m})$.

Then $f(t) - \varepsilon < h(t) < f(t) + \varepsilon$ for all $t \in K$. ∎